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On the 2-Token Graphs of Some Disjoint Union of Graphs

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Abstract

The k-token graph of a given graph G, is the graph which vertex set consists of all $k-$subsets of the vertex set of G and two vertices are connected by an edge exactly when their difference corresponds to an edge of G. In this paper, we give a description on the structure of the 2-token graph of disjoint union of multiple graphs. This result complements the previous findings regarding the properties of the k-token graphs.

Keywords: 2-token graph; disjoint union of graphs.

1 Introduction

All graphs discussed in this paper are finite, undirected, and simple, meaning they do not contain multiple edges or loops. In the context of any graph G, $V(G)$ and $E(G)$, are the vertex set and the edge set of G, respectively, and we write G as $G = (V(G), E(G))$. Any graph $H = (V(H), E(H))$ is defined as a subgraph of $G = (V(G), E(G))$ if and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore, if the subgraph H of G additionally satisfies the condition: for any $x, y \in V(H)$, if $xy \in E(G)$, then $xy \in E(H)$, it is termed an induced subgraph of G. In such instances, we say that H is induced by $V(H)$.

Numerous challenges in the fields of mathematics and computer science involve representing problems through the movement of objects on the nodes of a graph based on specific predefined rules. In the context of "graph pebbling," a single pebbling action entails removing two pebbles from one node and placing one pebble on an adjacent node. Theory on graph pebbling has been an interesting subject to study as evidenced on the survey given in reference [9]. It is also reported in [4] that the pebble motion problem on trees can be applied in many areas including memory management in distributed systems, robot motion planning, and deflection routing. For more particular development in graph pebbling one can also refer to [8]. Motivated by graph pebbling, Fabila-Monroy et al, as can be referred in [7], developed a new model which is called as a token graph.

As opposed to being constructed directly from an existing graph, graphs can also be created from algebraic structures like groups, rings, and lattices, establishing a meaningful link between algebra and graph theory. In [11], it is presented some results on the graph associated to a certain lattice, including the connectivity, the completeness, the hamiltonicity, the eulerianity, the matching number and the chromatic number. In [1], the authors investigated the properties of non-commuting graphs of groups. In this graph, two elements that are not in the centre of the group will be adjacent if and only if they are not commuting. Motivated by the work in [1], some non-commuting graphs of rings were introduced and investigated by the authors of reference [6]. For these two non-commuting graphs, various graph theoretical properties are obtained. In [2], it is introduced a bipartite graph constructed from a given group which vertex set is the union of the group (as the first part) and the set of all its subgroups (as the second part) and any element x of the group will be adjacent to a subgroup S if and only if $xS = Sx$, i.e. the left and the right cosets of x coincide. Some graph theoretical properties of the graph are presented, including the girth, the diameter and the dominating number. Specifically, for several finite groups some hamiltonicity and eulerianity properties of this bipartite graph are given in [13].

Given a graph $G = (V(G), E(G))$ of order $|V(G)| = n$. Let $k \leq n$ be a natural number and $P_k(V(G))$ be the set of all k-subsets of $V(G)$. According to [7], the k-token graph $\Gamma_k(G)$ of G is the graph with $P_k(V(G))$ as the vertex set and two arbitrary vertices $A, B \in P_k(V(G))$ are adjacent if and only if $A \triangle B = \{x, y\}$ for some $xy \in E(G)$ where $A \triangle B$ denotes the symmetric difference of A and B.

In [7], the authors provide insights into various properties of the k-token graph, including its connectivity, diameter, cliques, chromatic number, Hamiltonian paths, and Cartesian products. According to [7], the order and the size of the k-token graph of any graph G of order n and of size m are $\binom{n}{k}$ and $\binom{n-2}{k-1} m$, respectively. Moreover, it is also proved that the connectivity of graph G implies the connectivity of the k-token graph of G.

In the specific case of $k = 2$, more detailed information about the properties of the 2-token graphs can be found in [5]. This source offers essential and complete conditions for determining whether $\Gamma_2(G)$ is isomorphic to a cycle, as well as the conditions for a graph to be isomorphic to the 2-token graph of a star graph. Additionally, it outlines the necessary and sufficient conditions for the 2-token graph to be bipartite or a tree.

In the context of graph theory, it's important to remember that two graphs, $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, are considered isomorphic, denoted as $G_1 \cong G_2$, if there exists a bijection π from $V(G_1)$ to $V(G_2)$ that preserves adjacency in both directions. In other words, for any pair of vertices x and y in $V(G_1)$, $xy \in E(G_1)$ if and only if $\pi(x)\pi(y) \in E(G_2)$. This correspondence is referred to as an isomorphism. It's worth noting that the set of all isomorphisms for a graph G is a group respect to composition, and is called as the automorphism group of G, denoted as $Aut(G)$. There are some studies on the automorphism group of some particular graphs. In [3] the authors studied automorphism group of some power graphs over finite groups, while the authors in [14] presented automorphism groups of some bipartite graphs. In reference [12], automorphism groups of fundamental groups of graphs of groups in which the edge groups are incomparable up to conjugacy are presented. Automorphism groups for particular token graphs are given in [10] and particularly for the 2-token graphs,the automorphism groups are given in [17].

Recall that disjoint union of two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, denoted $G_1 \oplus G_2$, is the graph with vertex set,

$$V(G_1 \oplus G_2) = V(G_1) \cup V(G_2) \text{ and } E(G_1 \oplus G_2) = E(G_1) \cup E(G_2).$$

Visually, $G_1 \oplus G_2$ is a juxtaposition of G_1 and G_2. Clearly, operation $\oplus$ is binary, associative and commutative. Recall also that for any graph $K = (V(K), E(K))$ and $L = (V(L), E(L))$, the cartesian product graph of K and L, denoted $K\square L$, is the graph with vertex set,

$$V(K\square L) = V(K) \times V(L),$$

and any two different vertices (x, y) and (a, b) in $V(K) \times V(L)$, (x, y) and (a, b) are adjacent if and only if either $x = a$ and $yb \in E(L)$ or $xa \in E(K)$ and $y = b$. Clearly, $K\square L$ is isomorphic to $L\square K$ respect to a correspondence mapping any $(x, y) \in V(K\square L)$ to $(y, x) \in V(L\square K)$. Moreover, for any graphs G, K and H, it follows that $(G \oplus H)\square K = (G\square K) \oplus (H\square K)$. Observe, if G and K are two connected graphs, then $G\square K$ is also connected.

Consider a particular graph studied in [15] called a staircase graph (see also [16]). According to [15], the staircase graph SC_n is the graph with vertex set,

$$V(SC_n) = \{s_{i,j} | i = 0, 1, j = 0, 1, 2, \ldots, n\} \cup \{s_{i,j} | i = 2, \ldots, n, j = i - 1, \ldots, n\},$$

and $E(SC_n)$ which consists of all edges in the following forms:

$$\begin{array}{lll} s_{i,j}s_{i+1,j}, & i = 0, & 0 \le j \le n; \\ s_{i,j}s_{i+1,j}, & 1 \le i \le n-1, & i \le j \le n; \\ s_{i,j}s_{i,j+1}, & i = 0, 1, & 0 \le j \le n-1; \\ s_{i,j}s_{i,j+1}, & 2 \le i \le n, & i-1 \le j \le n-1. \end{array}$$

(See Figure 1).

Thus, we have $|V(SC_n)| = \frac{1}{2}(n+1)(n+2)$ and $|E(SC_n)| = n(n+3)$. In this paper, it will be shown that the 2-token graph $\Gamma_2(P_n)$ is isomorphic to SC_{n-3} with two additional pendant edges. Additionally, some insights into the structure of the 2-token graph for the disjoint union of multiple graphs are given, including the automorphism group of the 2-token graph G in the case where G comprises a disjoint union of several paths. Moreover, some illustrative examples for better understanding are also provided.

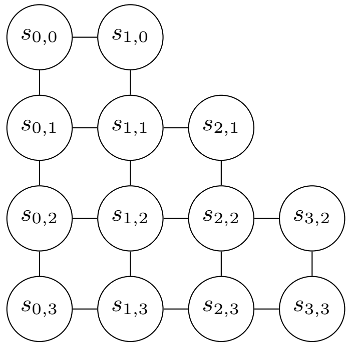

Figure 1: Staircase Graph SC_3.

2 Results

It's straightforward to notice that $\Gamma_2(C_3)$ is isomorphic to C_3, and $\Gamma_2(P_3)$ is isomorphic to P_3. In Figure 2 and Figure 3, we can see the disjoint union graph $P_3 \oplus C_3$ and its 2-token graph, respectively.

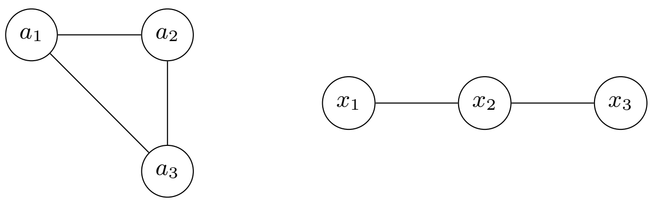

Figure 2: Disjoint union $C_3 \oplus P_3$ of C_3 and P_3.

Let $A = \{a_1, a_2\}$, $A_2 = \{a_1, a_3\}$, $A_3 = \{a_2, a_3\}$ and $B = \{x_1, x_2\}$, $B_2 = \{x_1, x_3\}$, $B_3 = \{x_2, x_3\}$. Let also $D_1 = \{a_1, x_1\}$, $D_2 = \{a_2, x_1\}$, $D_3 = \{a_3, x_1\}$, $D_4 = \{a_1, x_2\}$, $D_5 = \{a_2, x_2\}$, $D_6 = \{a_3, x_2\}$, $D_7 = \{a_1, x_3\}$, $D_8 = \{a_2, x_3\}$, $D_9 = \{a_3, x_3\}$. Then, the 2-token graph of $C_3 \oplus P_3$ can be described as shown in Figure 3.

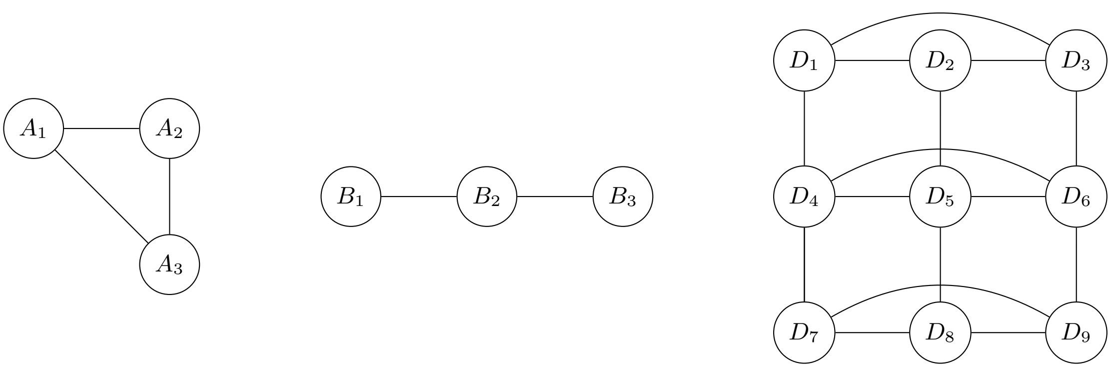

Figure 3: The 2-token graph $\Gamma_2(C_3 \oplus P_3)$ of $C_3 \oplus P_3$.

In this section, it is given a proof that the 2-token graph of disjoint union of two graphs consists of at least three components.

Lemma 2.1. *Given any two graphs K and H. If $G = K \oplus H$, then*

$$\Gamma_2(G) \cong \Gamma_2(K) \oplus \Gamma_2(H) \oplus (K \Box H).$$

Proof. It is clear that $\Gamma_2(K)$ and $\Gamma_2(H)$ are subgraphs of $\Gamma_2(G)$. Now, we will show that each vertex in $\Gamma_2(K)$ or in $\Gamma_2(H)$ is not connected by any single vertex in $\Gamma_2(G)$. Let,

$$V(K) = \{a_i | i = 1, 2, \dots, m\} \text{ and } V(H) = \{x_i | i = 1, 2, \dots, n\}.$$

Clearly, for any $A \in V(\Gamma_2(K))$ and for any $B \in V(\Gamma_2(H))$, it follows that $A \triangle B = \emptyset$ so that $AB \notin E(\Gamma_2(G))$. Secondly, let $W = V(\Gamma_2(G)) \setminus (V(\Gamma_2(K)) \cup V(\Gamma_2(H)))$. Clearly, any vertex $C \in W$ is of the form $A = \{a_i, x_j\}$ for some $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$. We will prove that any vertex in $V(\Gamma_2(K)) \cup V(\Gamma_2(H))$ is not adjacent to any vertex in $C \in W$. Let,

$$A \in (V(\Gamma_2(K)) \cup V(\Gamma_2(H))).$$

We will prove only when $A \in V(\Gamma_2(K))$.

For $A \in V(\Gamma_2(H))$, the proof is similar. Let, $A = \{a_p, a_q\} \in V(\Gamma_2(K))$ and $C = \{a_i, x_j\} \in W$ for some $p, q, i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$. Obviously, $AC \notin E(\Gamma_2(G))$ if $A \cap C = \emptyset$. If $A \cap C \neq \emptyset$, then $A \cap C = \{a_k\}$ for some $k \in \{p, q\}$. Hence, $A \triangle C$ is equal to $\{a_p, x_j\}$ or $\{a_q, x_j\}$. Therefore, $AC \notin E(\Gamma_2(G))$ as $a_p x_j, a_q x_j \notin E(G)$.

Now, consider the subgraph of $\Gamma_2(G)$ induced by W. Let $A, B \in W$ with $A \neq B$. Let $A = \{a_i, x_j\}$ and $B = \{a_k, x_l\}$. Clearly, $(i, j) \neq (k, l)$. We will consider three cases;

1. If $i = k$ and $j \neq l$, then $A \triangle B = \{x_j, x_l\}$. Thus, $AB \in E(\Gamma_2(G))$ if and only if $x_j x_l \in E(H)$.
2. If $i \neq k$ and $j = l$, then $A \triangle B = \{a_i, a_k\}$ and therefore $AB \in E(\Gamma_2(G))$ if and only if $a_i a_k \in E(K)$.
3. If $i \neq k$ and $j \neq l$, then clearly $AB \notin E(\Gamma_2(G))$ as $|A \triangle B| = 4$.

From these three cases, we have that $\{a_i, x_j\}\{a_k, x_l\} \in E(\Gamma_2(G))$ if and only if either $i = k$ and $x_j x_l \in E(H)$ or $a_i a_k \in E(K)$ and $j = l$.

Now, construct a mapping $f : W \to V(K) \times V(H)$ by $f(\{a_i, x_j\}) = (a_i, x_j)$ for any $\{a_i, x_j\} \in W$. It is easy to see that f is bijective. Moreover, by this correspondence we have that the subgraph induced by W in $\Gamma_2(G)$ is isomorphic to $K \Box H$. Therefore, we conclude $\Gamma_2(G) \cong \Gamma_2(K) \oplus \Gamma_2(H) \oplus (K \Box H)$. □

From Lemma 2.1, we can extend the result for arbitrary union disjoint graphs as given subsequently.

Theorem 2.1. *Given n graphs G_i, $i = 1, 2, \dots, n$, with $n \geq 2$. If $G = \bigoplus_{i=1}^{n} G_i$, then*

$$\Gamma_2(G) \cong \bigoplus_{i=1}^{n} \Gamma_2(G_i) \oplus \bigoplus_{1 \leq i < j \leq n} (G_i \Box G_j).$$

Proof. We will prove by mathematical induction on n. For $n = 2$, we have $G = G_1 \oplus G_2$ so that

$$\Gamma_2(G) = \Gamma_2(G_1 \oplus G_2) = \Gamma_2(G_1) \oplus \Gamma_2(G_2) \oplus (G_1 \Box G_2),$$

by Lemma 2.1. Thus, it is true for $n = 2$. Assume that the assertion is true for $n = k$. We will prove that the assertion is true for $n = k + 1$. Let $G = \bigoplus_{i=1}^{k+1} G_i$. Clearly, $G = (\bigoplus_{i=1}^{k} G_i) \oplus G_{k+1}$. Hence, by Lemma 2.1 and the induction hypothesis we have

$$\begin{aligned}
\Gamma_2(G) = \Gamma_2\left((\bigoplus_{i=1}^{k} G_i) \oplus G_{k+1}\right) &= \Gamma_2(\bigoplus_{i=1}^{k} G_i) \oplus \Gamma_2(G_{k+1}) \oplus \left((\bigoplus_{i=1}^{k} G_i)\Box G_{k+1}\right) \\
&= \bigoplus_{i=1}^{k} \Gamma_2(G_i) \oplus \bigoplus_{1\leq i<j\leq k} (G_i \Box G_j) \oplus \Gamma_2(G_{k+1}) \oplus \bigoplus_{i=1}^{k} (G_i \Box G_{k+1}) \\
&= \bigoplus_{i=1}^{k+1} \Gamma_2(G_i) \oplus \bigoplus_{1\leq i<j\leq k+1} (G_i \Box G_j).
\end{aligned}$$

We conclude that the assertion is true for any natural number $n \geq 2$. □

As an implication, we have this subsequent result.

Corollary 2.1. *Given connected graphs G_i's, $i = 1, 2, \ldots, n$. If $G = \bigoplus_{i=1}^{n} G_i$, then the 2-token graph $\Gamma_2(G)$ has precisely $n + \binom{n}{2}$ components.*

Proof. By Theorem 1 in [7], $\Gamma_2(G_i)$ is connected. Moreover, $G_i \Box G_j$ is also connected for any $i, j = 1, 2, \ldots, n$ and $i < j$. Therefore, by Theorem 2.1, the graph $\Gamma_2(\bigoplus_{i=1}^{n} G_i)$ has n components of the form $\Gamma_2(G_i)$, $i = 1, 2, \ldots, n$ and $\binom{n}{2}$ components in the form of cartesian products $G_i \Box G_j$, $i, j = 1, 2, \ldots, n$ and $i < j$. the assertion follows by Theorem 2.1. □

Now, let for any natural number $n \geq 2$, P_n be a path graph of order n. Observe that for $n = 3$, $P_3 \cong \Gamma_2(P_3)$. Now, let us see the following pictures of path graphs P_4 and P_5 with their corresponding 2-token graphs in Figures 4 and 5.

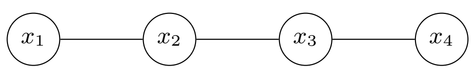

Figure 4: Path graph P_4.

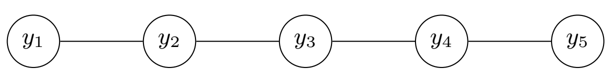

Figure 5: Path graph P_5.

From Figure 6 and Figure 7, we know that the 2-token graph of P_4 and of P_5 is isomorphic to some modification of staircase graph SC_1 and SC_2, each is modified by adding two pendants. As for $n = 3$, the 2-token graph P_n is isomorphic to P_n itself, we can summarize this fact in the following theorem.

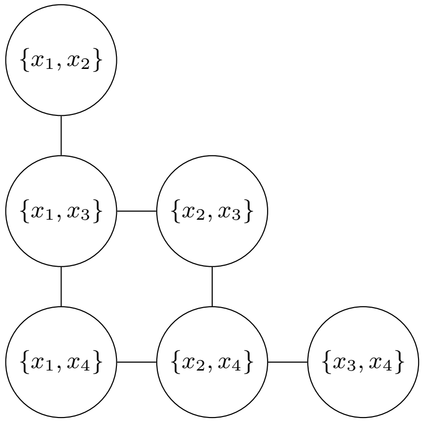

Figure 6: The 2-token graph $\Gamma_2(P_4)$.

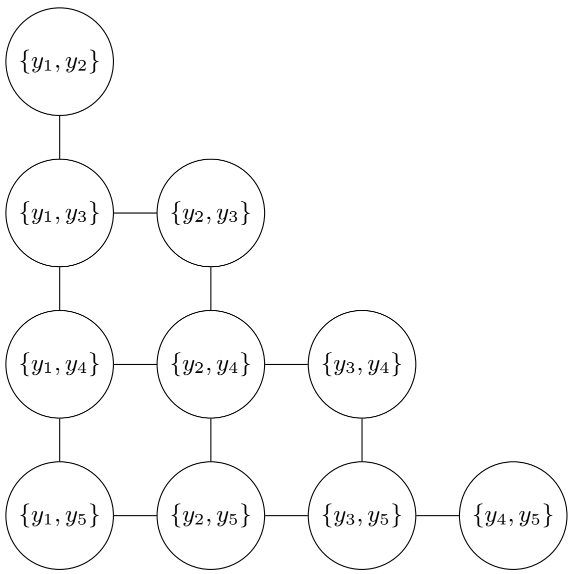

Figure 7: The 2-token graph $\Gamma_2(P_5)$.

Theorem 2.2. *Let P_n, $n \geq 4$ be a path graph. Then $\Gamma_2(P_n) \cong SC^{**}_{n-3}$ where SC^{**}_{n-3} is obtained from the SC_{n-3} by adding two additional vertices u and v and two pendants $us_{0,0}$ and $vs_{n-3,n-3}$.*

Proof. Let $V(P_n) = \{x_i | 1 \leq i \leq n\}$ and $E(P_n) = \{x_i x_{i+1} | 1 \leq i \leq n-1\}$. Then, for any $A = \{x_i, x_j\}, B = \{x_k, x_l\} \in V(\Gamma_2(P_n))$, $AB \in E(\Gamma_2(P_n))$ if and only if $A \triangle B = \{x_p, x_{p+1}\}$ for some $p \in \{i, j, k, l\}$. Let $u = \{x_1, x_2\}$ and $v = \{x_{n-1}, x_n\}$. Now, we construct a mapping $\psi : V(\Gamma_2(P_n)) \setminus \{u, v\} \to V(SC_{n-3})$ by $\psi(\{x_i, x_j\}) = s_{i-1,j-3}$, for any $i < j$ with $1 \leq i \leq n$ and $3 \leq j \leq n$. Let $\{x_i, x_j\}, \{x_k, x_l\} \in V(\Gamma_2(P_n)) \setminus \{u, v\}$. If $\{x_i, x_j\}\{x_k, x_l\} \in E(\Gamma_2(P_n))$, then we have $\{x_i, x_j\} \triangle \{x_k, x_l\} = \{x_r, x_{r+1}\}$ for some $x_r x_{r+1} \in E(P_n)$. WLOG, let $x_i = x_k$, i.e. $i = k$ and $j = r$, $l = r + 1$.

If $i < j$, then,

$$\psi(\{x_i, x_j\})\psi(\{x_k, x_l\}) = s_{i-1,j-3}s_{k-1,l-3} = s_{i-1,j-3}s_{i-1,l-3} = s_{i-1,r-3}s_{i-1,r-2} \in E(SC_{n-3}).$$

If $i > j$, then,

$$\psi(\{x_i, x_j\})\psi(\{x_k, x_l\}) = s_{i-3,j-1}s_{k-3,l-1} = s_{i-3,j-1}s_{i-3,l-1} = s_{i-3,r-1}s_{i-3,r} \in E(SC_{n-3}).$$

Conversely, let $\psi(\{x_i, x_j\})\psi(\{x_k, x_l\}) \in E(SC_{n-2})$. Let $i < j$ and $k < l$. Then,

$$s_{i-1,j-3}s_{k-1,l-3} \in E(SC_{n-3}).$$

It means that either $i = k$ and $l = j+1$ or $j = l$ and $k = i+1$. Thus, $\{x_k, x_l\} = \{x_i, x_{j+1}\}$ or $\{x_k, x_l\} = \{x_{i+1}, x_j\}$. Therefore, $\{x_i, x_j\}\{x_k, x_l\} \in E(\Gamma_2(P_n))$, so that ψ is an isomorphism between SC_{n-3} with the 2-token graph $\Gamma_2(P_n)$ without u and v, in the sense that u and v are removed from $\Gamma_2(P_n)$. These removals imply also the removals of edges $\{x_1, x_2\}\{x_2, x_3\}$ and $\{x_{n-2}, x_{n-1}\}\{x_{n-1}, x_n\}$. Therefore, by adding the two vertices u and v to SC_{n-3} and add two edges $us_{0,0}$ and $vs_{n-3,n-3}$ we will obtain the graph of size $(n-3)n + 2 = |E(\Gamma_2(P_n))|$. Furthermore, the resulted graph is isomorphic to the 2-token graph $\Gamma_2(P_n)$. □

From Theorem 2.2 we know that if f is an isomorphism on $\Gamma_2(P_n)$, $n \geq 4$ or $n = 3$, then f is either an identity element or a bijection switching vertices respect to "folding symmetry" of the graph. But particularly for $n = 4$, we have more isomorphisms on the graph. We formally summarize this fact in the subsequent theorem. For this, recall that the group $\mathbb{Z}_2$ is the additive group of all integer classes modulo 2. In the following lemma we prove that automorphism group of the 2-token graph of a path graph is either isomorphic to the group $\mathbb{Z}_2$ or the cartesian group $\mathbb{Z}_2 \times \mathbb{Z}_2$ under pointwise additive operation.

Lemma 2.2. *Let P_n, $n \geq 3$ be a path graph. Then,*

(i) $Aut(P_n) \cong Aut(\Gamma_2(P_n)) \cong \mathbb{Z}_2$ *for* $n = 3$.

(ii) $Aut(\Gamma_2(P_n)) \cong Aut(SC_1^{**}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *for* $n = 4$.

(iii) $Aut(P_n) \cong Aut(\Gamma_2(P_n)) \cong Aut(SC_{n-3}) \cong Aut(SC_{n-3}^{**}) \cong \mathbb{Z}_2$ *for* $n \geq 5$.

Proof. Let $V(P_n) = \{x_p | 1 \leq p \leq n\}$. For any n, $Aut(P_n)$ precisely consists of the identity mapping and the mapping $f : V(P_n) \to V(P_n)$ with $f(x_p) = n+1-p$ for all $p = 1, \ldots, n$. Therefore, $Aut(P_n) \cong \mathbb{Z}_2$,

(i) It is clear that $P_3 \cong \Gamma_2(P_3)$, so that $Aut(\Gamma_2(P_3)) \cong \mathbb{Z}_2$.

(ii) For $n = 4$, the 2-token graph $\Gamma_2(P_n) \cong SC_1^{**}$ is the graph given in Figure 6.
Let $A_1 = \{x_1, x_2\}$, $A_2 = \{x_1, x_3\}$, $A_3 = \{x_1, x_4\}$, $A_4 = \{x_2, x_3\}$, $A_5 = \{x_2, x_4\}$, $A_6 = \{x_3, x_4\}$.
The $Aut(\Gamma_2(P_4))$ consists of the identity mapping and the three mappings;

$$\theta_1 = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_1 & A_2 & A_4 & A_3 & A_5 & A_6 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_6 & A_5 & A_3 & A_4 & A_2 & A_1 \end{pmatrix}, \text{ and}$$
$$\theta_3 = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_6 & A_5 & A_4 & A_3 & A_2 & A_1 \end{pmatrix}.$$

It is straightforward that $Aut(\Gamma_2(P_n)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) For $n \geq 5$, let $E(P_n) = \{x_p x_{p+1} | 1 \leq p \leq n-1\}$. Let f be an isomorphism on $\Gamma_2(P_n)$. By Theorem 2.2, we have $\Gamma_2(P_n) \cong SC_{n-3}^{**}$. It is easy to see that f is either the identity mapping id

or a bijection f that maps $\{x_p, x_q\}$ to $\{x_{n+1-p}, x_{n+1-q}\}$ for every $\{x_p, x_q\} \in V(\Gamma_2(P_n))$. Obviously, $f^2 = f$. Hence, $Aut(\Gamma_2(P_n)) = \{id, f\}$, which is isomorphic to the group $\mathbb{Z}_2$. Hence, $Aut(\Gamma_2(P_n)) \cong Aut(SC^{**}_{n-3}) \cong \mathbb{Z}_2$. Moreover, f restricted on $V(SC_{n-3})$, i.e. $f|_{V(SC_{n-3})}$, is the only nonidentity isomorphism on SC_{n-3}. Also, $f|^2_{V(SC_{n-3})} = f|_{V(SC_{n-3})}$.

Thus, $Aut(SC_{n-3}) \cong \mathbb{Z}_2$, and as a consequence, we have

$$Aut(P_n) \cong Aut(\Gamma_2(P_n)) \cong Aut(SC_{n-3}) \cong Aut(SC^{**}_{n-3}) \cong \mathbb{Z}_2.$$

□

Let Sym_n, $n \geq 2$, be the symmetric group on $\{1, 2, \dots, n\}$ respect to the composition of functions. Recall that the dihedral D_{2n} is the subgroup of Sym_n representing all rigid motions on a regular n-gon. We have the following results.

Lemma 2.3. *Let P^i_n, $n \geq 3$, $i = 1, 2, \dots, k$ be path graphs of order n. Then,*

(*i*) $Aut(\Gamma_2(P^1_n) \oplus \dots \oplus \Gamma_2(P^k_n)) \cong Sym_k \times \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{k\text{-times}}$ *for* $n \neq 4$.

(*ii*) $Aut(\Gamma_2(P^1_n) \oplus \dots \oplus \Gamma_2(P^k_n)) \cong Sym_k \times \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{2k\text{-times}}$ *for* $n = 4$.

Proof. For all $i = 1, \dots, k$, let $V(P^i_n) = \{x^i_j | j = 1, \dots, n\}$ and $E(P^i_n) = \{x^i_j x^i_{j+1} | j = 1, \dots, n-1\}$. Then, we obtain $V(\Gamma_2(P^i_n)) = \{\{x^i_p, x^i_q\} | p < q \text{ and } p, q = 1, \dots, n\}$ for each $i = 1, \dots, k$. Now, let $G = \Gamma_2(P^1_n) \oplus \dots \oplus \Gamma_2(P^k_n)$. Let $\alpha \in Sym_k$ be arbitrary. We construct

$$\theta_\alpha : \{V(\Gamma_2(P^i_n)) | i = 1, \dots, k\} \to \{V(\Gamma_2(P^i_n)) | i = 1, \dots, k\},$$

with $\theta_\alpha(V(\Gamma_2(P^i_n))) = V(\Gamma_2(P^{\alpha(i)}_n))$. Let $n \neq 4$. Let $(\alpha, f_1, \dots, f_k)$ be arbitrary $(k+1)$-tuple with $f_i : \Gamma_2(P^{\alpha(i)}_n) \to \Gamma_2(P^{\alpha(i)}_n)$ is either the identity mapping $id_{\Gamma_2(P^{\alpha(i)}_n)}$ on $\Gamma_2(P^{\alpha(i)}_n)$ or a mapping defined by $f_i(\{x^{\alpha(i)}_p, x^{\alpha(i)}_q\}) = \{x^{\alpha(i)}_{n+1-q}, x^{\alpha(i)}_{n+1-p}\}$ for every $\{x^{\alpha(i)}_p, x^{\alpha(i)}_q\} \in V(\Gamma_2(P^{\alpha(i)}_n))$, for all $i = 1, \dots, k$. These $(k+1)-$tuples define isomorphisms on G for any $\alpha \in Sym_k$. Conversely, as $n \neq 4$, by Lemma 2.2, each isomorphism on G belongs to $Sym_k \times \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{k\text{-times}}$ so that,

$$Aut(G) = Aut(\Gamma_2(P^1_n) \oplus \dots \oplus \Gamma_2(P^k_n)) \cong Sym_k \times \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{k\text{-times}}.$$

For $n = 4$, by Lemma 2.2, $\Gamma_2(P_n) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by a similar argument, we have

$$Aut(\Gamma_2(P^1_n) \oplus \dots \oplus \Gamma_2(P^k_n)) \cong Sym_k \times \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{2k\text{-times}}.$$

□

We close this section with the following discussion on the 2-token graphs of disjoint union of path graphs.

Theorem 2.3. *Let P^i_n, $n \geq 3$, $i = 1, 2, \dots, k$ be path graphs. Let $p = \dfrac{k(k-1)}{2}$. If $G = P^1_n \oplus \dots \oplus P^k_n$, then,*

$$(i)\quad Aut(G) \cong Sym_k \times \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{k\text{-times}} \times Sym_p \times \underbrace{D_8 \times \ldots \times D_8}_{p\text{-times}}, \textit{for } n \neq 4.$$

$$(ii)\quad Aut(G) \cong Sym_k \times \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{2k\text{-times}} \times Sym_p \times \underbrace{D_8 \times \ldots \times D_8}_{p\text{-times}}, \textit{for } n = 4.$$

Proof. Let $G = P_n^1 \oplus \ldots \oplus P_n^k$. Then, by Theorem 2.1 we have,

$$\Gamma_2(G) \cong \bigoplus_{i=1}^{k} \Gamma_2(P_n^i) \oplus \bigoplus_{1 \le i < j \le k} (P_n^i \square P_n^j).$$

Clearly, $\bigoplus_{i=1}^{k} \Gamma_2(P_n^i)$ and $\bigoplus_{1 \le i<j \le k}(P_n^i \square P_n^j)$ are invariant under any isomorphism on $\Gamma_2(G)$. It is also obvious that $Aut(P_n^i \square P_n^j) \cong D_8$. Moreover, we have also $\dfrac{k(k-1)}{2}$ disjoint union of cartesian product graphs in the form $P_n^i \square P_n^j$. Similar to the proof of Lemma 2.3, we can permute the vertex sets of these $p = \dfrac{k(k-1)}{2}$ graphs, so that we have,

$$Aut(\bigoplus_{1 \le i<j \le k} (P_n^i \square P_n^j)) \cong Sym_p \times \underbrace{D_8 \times \ldots \times D_8}_{p\text{-times}}.$$

Altogether by Lemma 2.3, we have,

$$Aut(G) \cong Sym_k \times \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{k\text{-times}} \times Sym_p \times \underbrace{D_8 \times \ldots \times D_8}_{p\text{-times}},$$

for $n \neq 4$ and

$$Aut(G) \cong Sym_k \times \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{2k\text{-times}} \times Sym_p \times \underbrace{D_8 \times \ldots \times D_8}_{p\text{-times}},$$

for $n = 4$. This completes the proof. □

We close this section by presenting Figure 8, as an illustration of the k-token graph of the disjoint union $P_5^1 \oplus P_5^2 \oplus P_5^3$ of path graphs P_5^i, for $i = 1, 2, 3$.

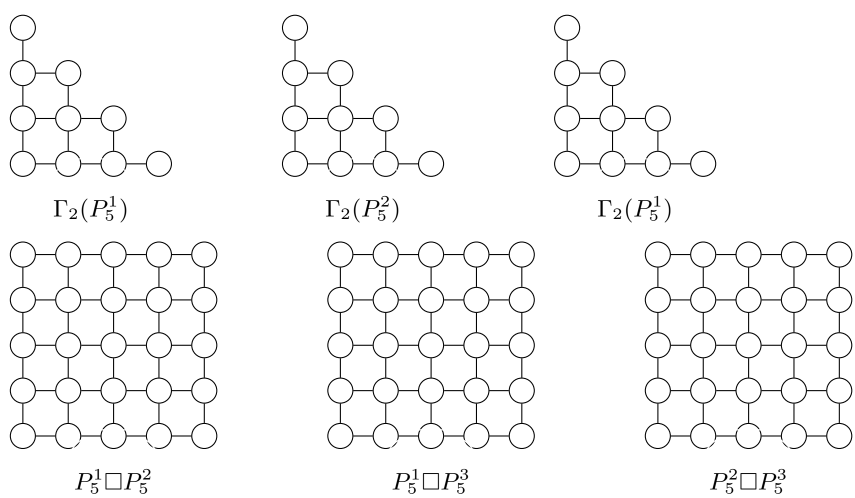

Figure 8: The 2-token graph $\Gamma_2(P_5^1 \oplus P_5^2 \oplus P_5^3)$.

3 Conclusions

From our earlier discussion, we have obtained the structure of the 2-token graph of disjoint union of multiple graphs. Nevertheless, for any given k, the structure of the k-token graph of disjoint union of multiple graphs is not yet ascertained. Additionally, the structures of the k-token graph of graphs constructed by performing some graph operations in general are also still unknown. These questions present intriguing areas for future research.

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